## 5.2 The Characteristic Equation

We will use an example to introduce what is a **characteristic equation** for a given matrix $A$. The solutions for such an equation are the eigenvalues of the matrix.

**Example 1.** Find the eigenvalues of $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$.

ANS: By def. we need to find the scalars $\lambda$ such that

$$A\vec{x} = \lambda\vec{x} \iff (A-\lambda I)\vec{x} = \vec{0}$$

has nontrivial solution. By the invertible matrix theorem, this is equivalent to finding $\lambda$ such that

$$\boxed{|A-\lambda I| = 0}$$

So we solve the equation $|A-\lambda I| = 0$

$$|A-\lambda I| = \left| \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 9 = \lambda^2 - 10\lambda + 16 = 0$$

$\Rightarrow (\lambda-2)(\lambda-8) = 0 \Rightarrow \lambda = 2$ or $\lambda = 8$.

Thus $\lambda = 2$ and $8$ are the eigenvalues.

On the next page, we define the equation

$$|A-\lambda I| = 0 \text{ or } \det(A-\lambda I) = 0$$

as the characteristic equation for $A$.

We review the property of determinants below:

**Theorem 3. Properties of Determinants**
Let $A$ and $B$ be $n \times n$ matrices.

a. $A$ is invertible if and only if $\det A \neq 0$.

b. $\det AB = (\det A)(\det B)$.

c. $\det A^T = \det A$.

d. If $A$ is triangular, then $\det A$ is the product of the entries on the main diagonal of $A$.

e. A row replacement operation on $A$ does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

**Theorem. The Invertible Matrix Theorem (continued)**
Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if and only if
r. The number 0 is not an eigenvalue of $A$.

### The Characteristic Equation

The scalar equation $\det(A - \lambda I) = 0$ in Example 1 is called the **characteristic equation** of $A$. From the argument of Example 1, we have the following fact:

A scalar $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$ if and only if $\lambda$ satisfies the characteristic equation

$$\det(A - \lambda I) = 0$$

**Example 2.** Find the $\underbrace{\text{characteristic polynomial}}_{}$ of each matrix using expansion across a row or down a column. [Note: Finding the characteristic polynomial of a $3 \times 3$ matrix is not easy to do with just row operations, because the variable $\lambda$ is involved.]

det(A-λI)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -3 & 6 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\left( \begin{bmatrix} 1 & 0 & 1 \\ -3 & 6 & 1 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right)$$

$$= \det\left( \begin{bmatrix} 1-\lambda & 0 & 1 \\ -3 & 6-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{bmatrix} \right)$$

Make a cofactor expansion along the third row

$$= (4-\lambda)\cdot(-1)^{3+3}\det\left(\begin{bmatrix}1-\lambda & 0\\ -3 & 6-\lambda\end{bmatrix}\right)$$

$$= (4-\lambda)(1-\lambda)(6-\lambda)$$

$$= -\lambda^3+11\lambda^2-34\lambda+24$$

**Example 3.** For the given matrix, list the eigenvalues, repeated according to their multiplicities.

$$A = \begin{bmatrix} 5 & 0 & 0 & 0\\ 8 & -4 & 0 & 0\\ 0 & 7 & 1 & 0\\ 1 & -5 & 2 & 1\end{bmatrix}$$

You can use Thm 1 in §5.1. The eigenvalues are

5, -4, 1, 1.

Or we compute

$$|A-\lambda I| = \begin{vmatrix} 5-\lambda & 0 & 0 & 0\\ 8 & -4-\lambda & 0 & 0\\ 0 & 7 & 1-\lambda & 0\\ 1 & -5 & 2 & 1-\lambda\end{vmatrix} = (5-\lambda)(-4-\lambda)(1-\lambda)(1-\lambda) = 0$$

$$\Rightarrow \lambda = 5, -4, 1, 1.$$

**Similarity**

If $A$ and $B$ are $n \times n$ matrices, then $A$ **is similar to** $B$ if there is an invertible matrix $P$ such that $P^{-1}AP = B$, or, equivalently, $A = PBP^{-1}$.

Remark: $\cancel{P}P^{-1}A\cancel{P}\cancel{P^{-1}} = PBP^{-1} \Rightarrow A = PBP^{-1}$

- Writing $Q$ for $P^{-1}$, we have $Q^{-1}BQ = A$. So $B$ is also similar to $A$, and we say simply that $A$ and $B$ **are similar**.
- Changing $A$ into $P^{-1}AP$ is called a **similarity transformation.**

**Theorem 4.**

If $n \times n$ matrices $A$ and $B$ are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

**Warnings:**

1. The matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar even though they have the same eigenvalues.

2. Similarity is not the same as row equivalence. (If $A$ is row equivalent to $B$, then $B = EA$ for some invertible matrix $E$.) Row operations on a matrix usually change its eigenvalues.

**Example 4.** Show that if $A$ and $B$ are similar, then $\det A = \det B$.

ANS: If $A$ and $B$ are similar, then by def

$$P^{-1}AP = B$$

$$\xRightarrow[\text{both sides}]{\text{Take det}} \det(P^{-1}AP) = \det B$$

$$\Longrightarrow \det(P^{-1}) \det A \det P = \det B$$

$$\xRightarrow{\det(P^{-1}) = \frac{1}{\det P}} \frac{1}{\det P} \cdot \det P \det A = \det B$$

$$\Longrightarrow \det A = \det B.$$

**Exercise 5.**

It can be shown that the algebraic multiplicity of an eigenvalue $\lambda$ is always greater than or equal to the dimension of the eigenspace corresponding to $\lambda$. Find $k$ in the matrix $A$ below such that the eigenspace for $\lambda = 3$ is two-dimensional:

$$A = \begin{bmatrix} 3 & -2 & 4 & -1 \\ 0 & 5 & k & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ANS: Recall the eigenspace is the nullspace of the matrix $A-\lambda I$. We reduce the augmented matrix for the equation $(A-\lambda I)\vec{x} = \vec{0}$

Note $A-3I = \begin{bmatrix} 0 & -2 & 4 & -1 \\ 0 & 2 & k & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

$[A-3I \quad \vec{0}]$

$$= \left[\begin{array}{cccc|c} 0 & -2 & 4 & -1 & 0 \\ 0 & 2 & k & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 0 & 2 & k & 0 & 0 \\ 0 & -2 & 4 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 0 & 2 & k & 0 & 0 \\ 0 & 0 & 4+k\,? & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

If the eigenspace is 2-dimensional, we need two free varibles for the corresponding system.

This happens if and only if $4+k=0$

i.e. $k=-4$.